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INVARIANTS OF LIE SUPERALGEBRAS ACTING ON ASSOCIATIVE ALGEBRAS*

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ABSTRACT

Let R be a semiprime algebra over a field K acted on by a finite-dimensional Lie superalgebra L. The purpose of this paper is to prove a series of going-up results showing how the structure of the subalgebra of invariants R^L is related to that of R. Combining several of our main results we have:

THEOREM: *Let R be a semiprime K-algebra acted on by a finite-dimensional nilpotent Lie superalgebra L such that if characteristic $K = p$ then L is restricted and if characteristic $K = 0$ then L acts on R as algebraic derivations and algebraic superderivations.*

(i) *If R^L is right Noetherian, then R is a Noetherian right R^L-module. In particular, R is right Noetherian and is a finitely generated right R^L-module.*

(ii) *If R^L is right Artinian, then R is an Artinian right R^L-module. In particular, R is right Artinian and is a finitely generated right R^L-module.*

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(iii) *If R^L is finite-dimensional over K then R is also finite-dimensional over K.*

(iv) *If R^L has finite Goldie dimension as a right R^L-module, then R has finite Goldie dimension as a right R-module.*

(v) *If R^L has Krull dimension α as a right R^L-module, then R has Krull dimension α as a right R^L-module. Thus R has Krull dimension at most α as a right R-module.*

(vi) *If R is prime and R^L is central, then R satisfies a polynomial identity.*

(vii) *If L is a Lie algebra and R^L is central, then R satisfies a polynomial identity.*

We also provide counterexamples to many questions which arise in view of the results in this paper.

1. Introduction

Let R be a semiprime algebra over a field K acted on by a finite-dimensional Lie superalgebra L. The purpose of this paper is to prove a series of going-up results showing how the structure of the subalgebra of invariants R^L is related to that of R. All of our Lie superalgebras will be nilpotent and spanned by derivations and superderivations which are algebraic over K, when viewed as K-linear transformations of R. Before introducing the definitions and terminology that will be use throughout this paper, we begin with an example which will put our results into the proper perspective. The example is based on one by Bergman-Kharchenko [P, Chapter 6] on group actions. Let $S = K[x,y]$ be the noncommutative free algebra over K in 2 variables and let $R = S_2$, the 2×2 matrices over S. Next, let L be the 3-dimensional solvable Lie algebra of inner derivations of R spanned by the derivations induced by commutation by the elements $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$, and $e_3 = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$. Note that all the derivations in L are algebraic as K-linear transformations of R and if characteristic $K = p$, then L is restricted. The inner derivations induced by e_2 and e_3 span a 2-dimensional abelian ideal I of L and I is also restricted in the characteristic p case. The subalgebra of invariants R^I under the action of I is the commutative subalgebra consisting of all matrices of the form $\begin{pmatrix} \alpha & b \\ 0 & \alpha \end{pmatrix}$, where $\alpha \in K$ and $b \in S$. Furthermore, if a matrix of the form $\begin{pmatrix} \alpha & b \\ 0 & \alpha \end{pmatrix}$ also commutes with e_1, then $b = 0$. Hence the subalgebra of invariants R^L is isomorphic to the field K. Clearly R^L satisfies many ring theoretic properties such as being Noetherian, Artinian, satisfying a polynomial identity, and being finite-dimensional over K, whereas R has none of these properties. In addition, no proper ideal of R intersects R^L nontrivially. Therefore, even if we assume that R is prime and L is solvable of relatively small dimension, we cannot hope to prove going-up results for R^L and R.

Let us analyze this example more closely. We say that a derivation d of a K-algebra R is separable if its minimum polynomial is of the form $\alpha_n d^n + \cdots + d$, where $\alpha_i \in K$. The invariants of separable derivations have been studied using the techniques of group-graded rings [BC] and many going-up results hold, provided that R is semiprime. The results on group-graded rings that we will need in this paper appear in Proposition 3.1. However, if an algebra with a separable derivation is not semiprime, then the going-up results in Proposition 3.1 do not hold. This occurs in our example where the derivation d induced by e_1 acts on R^I such that d satisfies the polynomial $d^3 - d$ and R^I has a nilpotent ideal of codimension 1. In our example, we can consider R as first being acted on by the nilpotent derivations in I and then by the separable derivation d which can be thought of as an element of the quotient Lie algebra L/I. On the other hand, now suppose we were given a Lie algebra L which can be decomposed in the opposite way. That is, suppose L contained an ideal I such that I was spanned by separable derivations and the quotient L/I consisted of nilpotent derivations. In Theorem 2.3, we show that even if there are no additional hypotheses on an algebra, various chain conditions can be lifted up from the invariants provided the action is by nilpotent derivations. Thus in this case, we could prove going-up results from R^L to R^I. We could then apply Proposition 3.1 to the action of I on R to lift the chain conditions from R^I to R. Although this is not possible in the solvable case, we will show that it does work in the nilpotent case, even for the action of Lie superalgebras. This will lead us to our first main result of this paper:

THEOREM 3.3: *Let R be a semiprime K-algebra acted on by a finite-dimensional nilpotent Lie superalgebra L such that if characteristic $K = p$ then L is restricted and if characteristic $K = 0$ then L acts on R as algebraic derivations and algebraic superderivations.*

(i) *If R^L is right Noetherian, then R is a Noetherian right R^L-module. In par-*

ticular, R is right Noetherian and is a finitely generated right R^L-module.

(ii) *If R^L is right Artinian, then R is an Artinian right R^L-module. In particular, R is right Artinian and is a finitely generated right R^L-module.*

(iii) *If R^L is finite-dimensional over K then R is also finite-dimensional over K.*

(iv) *If R^L has finite Goldie dimension as a right R^L-module, then R has finite Goldie dimension as a right R-module.*

(v) *If R^L has Krull dimension α as a right R^L-module, then R has Krull dimension α as a right R^L-module. Thus R has Krull dimension at most α as a right R-module.*

We can put Theorem 3.3 and some of the examples in Section 5 into better perspective by looking at their relationship with some previous results. In [G] and [GM] semiprime rings R and their invariants $R^{(d)}$ under the actions of algebraic derivations d are studied. In [GM] it is shown that R is Goldie if and only if $R^{(d)}$ is Goldie, and in [G] it is shown that R is Artinian if and only if $R^{(d)}$ is Artinian. Theorem 3.3 shows that the going-up portion of results similar to those in [G] and [GM] can be extended from a single derivation to a finite-dimensional nilpotent Lie superalgebra. However, in Section 5, we show that the going-down portion of the results in [G] and [GM] cannot be extended to the action of two commuting derivations. In particular, in Example 5.2, we give an example where R is simple Artinian and L is a 2-dimensional abelian Lie algebra, yet R^L is neither Artinian nor Goldie. Theorem 3.3 includes the going-up part of the Noetherian analog of the results in [G] and [GM]. However, in Example 5.1, we see that the going-down part does not hold as we provide an example where R is prime and left Noetherian, but $R^{(d)}$ is not left Noetherian. In addition, in Example 5.4, we show that the results in Theorem 3.3 on the finite generation of R over R^L require that R^L satisfy some chain condition, even if we assume that R is prime and L is spanned by a single derivation.

If one makes additional assumptions about the structure of R, one can obtain results for the actions of more general Lie algebras. For example, it is shown in [B2], that if R has no nilpotent elements and if L is a finite-dimensional restricted solvable Lie algebra then R is Goldie if and only if R^L is Goldie and R satisfies a polynomial identity if and only of R^L satisfies a polynomial identity. Furthermore, it is shown in [B4], that if R is a domain then the above results hold even without assuming that L is solvable.

Returning briefly to the Bergman-Kharchenko example, we observe that the ideal I is an abelian Lie algebra such that the invariants R^I are commutative, yet R does not satisfy a polynomial identity. Thus, even in the abelian case, the property of satisfying a polynomial identity does not lift from the invariants to all of R. However, if the invariants are central, we obtain one of the two main results of Section 4:

THEOREM 4.3: *Let R be a prime K-algebra acted on by a finite-dimensional nilpotent Lie superalgebra L such that if characteristic $K = p$ then L is restricted and if characteristic $K = 0$ then L acts on R as algebraic derivations and algebraic superderivations. If R^L is central, then R satisfies a polynomial identity.*

In Section 4, we also extend Theorem 4.3 to the semiprime case when L is a Lie algebra. Finally, Section 5 contains counterexamples to several questions which arise in light of the results in Sections 2, 3, and 4.

We can now introduce the definitions and terminology that we will use throughout this paper. If R is an algebra over a field K, let $\mathrm{End}_K(R)$ denote the K-linear maps from R to R. If σ is a K-linear automorphism of R, let $D_0 = \{\delta \in \mathrm{End}_K(R)\colon \delta(rs) = \delta(r)s + r\delta(s)$ and $\delta\sigma(r) = \sigma\delta(r)$, for all $r, s \in R\}$ and $D_1 = \{\delta \in \mathrm{End}_K(R)\colon \delta(rs) = \delta(r)s + \sigma(r)\delta(s)$ and $\delta\sigma(r) = -\sigma\delta(r)$, for all $r, s \in R\}$. If $\sigma^2 = 1$ and characteristic $K \neq 2$, then $D_0 \oplus D_1$ is a Lie superalgebra and the elements of D_0 and D_1 are respectively, derivations and superderivations of R. The superbracket on $D_0 \oplus D_1$ is defined as $[\delta_1, \delta_2] = \delta_1\delta_2 - (-1)^{ij}\delta_2\delta_1$, where $\delta_1 \in D_i$, $\delta_2 \in D_j$ and $i, j \in \{0, 1\}$. If $L = L_0 \oplus L_1$ is a Lie superalgebra, we say that L acts on R if there is a homomorphism of Lie superalgebras $\psi\colon L \to D_0 \oplus D_1$, where $\psi(L_i) \subseteq D_i$, for $i = 0, 1$. When there is no confusion, we may simply assume that $L \subseteq D_0 \oplus D_1$. In particular, we will often identify the elements of L_0 and L_1 with their images under ψ and refer to them as derivations and superderivations. All of our Lie superalgebras will be assumed to be finite-dimensional over K. However, to study the relationship between R^L and R, we will need the additional assumption that the derivations and superderivations from L_0 and L_1 are algebraic over K when viewed as elements of $\mathrm{End}_K(R)$. In the characteristic p case, this is equivalent to assuming that L is restricted and that ψ also satisfies $\psi(l^{[p]}) = \psi(l)^p$, where $[p]$ is the pth power map and $l \in L_0$. Note that when $L_1 \neq 0$, we must assume that the characteristic of K is not equal to 2. Additional properties of restricted Lie superalgebras can be found in [B3]. In the characteristic 0 case, we will need to explicitly state that L acts as algebraic derivations and superderivations. There is an interesting difference between derivations and superderivations in the characteristic 0 case. If R is semiprime of characteristic 0, then all algebraic derivations of R become inner when extended to the Martindale quotient ring of R. However, this is not the case for superderivations, thus the study of the invariants of superderivations cannot be reduced down to the study of centralizers of subalgebras.

Another way to view the condition that the derivations from L_0 and L_1 are algebraic is that ψ induces an associative homomorphism from the universal enveloping algebra $U(L)$ to $\mathrm{End}_K(R)$ and its image is finite-dimensional if and only if the derivations and superderivations from L_0 and L_1 are algebraic. The enveloping algebra $U(L)$ is not a Hopf algebra when L is not a Lie algebra. However, in this case we can view the automorphism σ as also acting on $U(L)$ by acting as the identity on L_0 and by negating the elements of L_1. Letting G be the group $\{1, \sigma\}$, we can form the skew group ring $H = U(L) * G$ and H is now a Hopf algebra acting R. This Hopf algebra and the smash product $R\#H$ will be used in Section 4 to extend the results in [BCF] on central rings of invariants and polynomial identities. For more details on the construction of H, we refer to [B3] and for more details on $R\#H$ and how R is a left $R\#H$-module, we refer to [BCF]. In the restricted case, we can replace $U(L)$ and $U(L) * G$ by $u(L)$ and $u(L) * G$, where $u(L)$ is the restricted enveloping algebra of L. In Sections 3 and 4, we will use the fact that both $u(L)$ and $u(L) * G$ are finite-dimensional. All of the results on Lie superalgebras can be specialized to the Lie algebra case by ignoring the presence of G and σ and viewing ψ as a Lie homomorphism from L to the K-linear derivations of R. In addition, when L is a Lie algebra, we allow the characteristic of K to be 2.

When L acts on R, we define the subalgebra of invariants R^L to be $\{r \in R\colon \delta(r) = 0, \text{ for all } \delta \in \psi(L)\}$. In particular, if $\delta \in \mathrm{End}_K(R)$, we let $R^{(\delta)}$ denote the set $\{r \in R\colon \delta(r) = 0\}$. If I is a Lie superideal of L, then the quotient Lie superalgebra L/I acts on the subalgebra R^I such that $R^L = (R^I)^{L/I}$. Depending upon the context, the symbol $[\ ,\ ]$ may represent either the superbracket map sending $L \times L$ to L or the commutator map $[a, b] = ab - ba$, where a, b belong to an associative algebra. Inductively, we let $L(1) = L$ and $L(n+1) = [L(n), L]$ and we say that L is nilpotent if there exists a positive integer N such that $L(N) = 0$. If A is an associative algebra or a Lie algebra we will let $Z(A)$ denote its center. Finally, all ideals of associative algebras will be assumed to be two-sided, unless it is explicitly stated otherwise.

2. Nilpotent derivations, superderivations, and skew derivations

In this section we discuss the important special case where all the elements of L_0 and L_1 act as nilpotent derivations and superderivations. We also obtain a result on nilpotent skew derivations which may be of independent interest. All the arguments used in this section are module-theoretic in nature. Thus the results we obtain do not require any hypotheses on the structure of either our algebras R or our Lie superalgebras L, other than L being finite-dimensional. Of fundamental importance in this section are the following well-known facts about modules which we state without proof.

LEMMA 2.1: *Let $\phi: U \to V$ be a homomorphism of right A-modules with kernel W, where A is a K-algebra.*

(i) *If V and W are Noetherian right A-modules, then so is U.*
(ii) *If V and W are Artinian right A-modules, then so is U.*
(iii) *If V and W are finite-dimensional over K, then $\dim_K U \leq \dim_K V + \dim_K W$.*
(iv) *If V and W have finite right Goldie dimension over A, then so does U.*
(v) *If V and W have right Krull dimension, then U has right Krull dimension and $K\dim\ U_A \leq \sup(K\dim\ V_A, K\dim\ W_A)$.*

If $\delta \in \mathrm{End}_K(R)$ then δ is a skew derivation of R if there exists a K-linear automorphism τ of R such that $\delta(rs) = \delta(r)s + \tau(r)\delta(s)$, for all $r, s \in R$. Therefore if $a \in R^{(\delta)}, r \in R$ we have $\delta(ra) = \delta(r)a$. Hence, δ is a right $R^{(\delta)}$-module map. We use this observation to prove the following result on the invariants of nilpotent skew derivations:

PROPOSITION 2.2: *Suppose δ is a nilpotent skew derivation of a K-algebra R and let $R^{(\delta)} = \{r \in R: \delta(r) = 0\}$.*

(i) *If $R^{(\delta)}$ is right Noetherian, then R is a Noetherian right $R^{(\delta)}$-module. In particular, R is right Noetherian and is a finitely generated right $R^{(\delta)}$-module.*
(ii) *If $R^{(\delta)}$ is right Artinian, then R is an Artinian right $R^{(\delta)}$-module. In particular, R is right Artinian and is a finitely generated right $R^{(\delta)}$-module.*
(iii) *If $R^{(\delta)}$ is m-dimensional over K then R has dimension at most mn, where $\delta^n(R) = 0$.*

(iv) *If $R^{(\delta)}$ has finite Goldie dimension as a right $R^{(\delta)}$-module, then R has finite Goldie dimension as a right $R^{(\delta)}$-module.*

(v) *If A is a subring of $R^{(\delta)}$ such that $R^{(\delta)}$ has Krull dimension α as a right A-module, then R has Krull dimension α as a right A-module.*

Proof: If $\delta^n(R) = 0$, let $R_i = \{r \in R\colon \delta^i(r) = 0\}$ for $i \leq n$. Then $R = R_n$, $R^{(\delta)} = R_1$, and we have the sequence of right $R^{(\delta)}$-modules

$$R = R_n \to R_{n-1} \to \cdots \to R_2 \to R_1 = R^{(\delta)}.$$

For $2 \leq i \leq n$, we have $\delta(R_i) \subseteq R_{i-1}$ and $R^{(\delta)}$ is the kernel at every stage of the sequence. Most of (i)–(v) follows by applying $n-1$ times the corresponding parts of Lemma 2.1 to the sequence of $R^{(\delta)}$-modules starting with $R^{(\delta)}$ and finishing with R. There are a few remaining details needed to complete the proofs of (i), (ii), and (v). For (i), since R is Noetherian as a right $R^{(\delta)}$-module, it must be finitely generated and R must be a right Noetherian ring. For (ii), $R^{(\delta)}$ is a right Artinian ring with unit, hence is a right Noetherian ring. Thus by applying (i), R is finitely generated over the right Artinian ring $R^{(\delta)}$, hence R is an Artinian $R^{(\delta)}$-module and must be a right Artinian ring. For (v), repeated use of Lemma 2.1(v) yields $K\dim\ R_A \leq K\dim\ R^{(\delta)}_A$. However, since $R^{(\delta)} \subseteq R$, we have $K\dim\ R_A = K\dim\ R^{(\delta)}_A$. ∎

In [Ka, Theorem 11] , it is shown that if an associative algebra A contains a Lie subset S of nilpotent elements such that S spans a finite-dimensional subspace of A, then S is associative-nilpotent. The argument, which is due to Jacobson, can be adapted in a straightforward way to handle the case where S is no longer a Lie subset of A, but does have the property that for any $a, b \in S$ there exists a scalar $\alpha = \alpha(a, b) \in K$ such that $ab - \alpha ba \in S$.

If L is a Lie superalgebra of nilpotent derivations and superderivations of a K-algebra R, we can apply the above to our situation by letting $A = \mathrm{End}_K(R)$ and $S = \psi(L_0) \cup \psi(L_1)$. Thus even if L is not nilpotent, $\psi(L_0) \cup \psi(L_1)$ is an associative-nilpotent subset of $\mathrm{End}_K(R)$. Therefore $\psi(L)$ is certainly nilpotent as a Lie superalgebra. As a result, in the following theorem we do not need to assume that L is nilpotent, since its image in $\mathrm{End}_K(R)$ is a nilpotent superalgebra.

THEOREM 2.3: *Let R be a K-algebra acted on by a finite-dimensional Lie superalgebra L of nilpotent derivations and superderivations and let $R^L = \{r \in R\colon \delta(r) = 0$, for all $\delta \in \psi(L)\}$.*

(i) *If R^L is right Noetherian, then R is a Noetherian right R^L-module. In particular, R is right Noetherian and is a finitely generated right R^L-module.*
(ii) *If R^L is right Artinian, then R is an Artinian right R^L-module. In particular, R is right Artinian and is a finitely generated right R^L-module.*
(iii) *If R^L is finite-dimensional over K then R is also finite dimensional over K.*
(iv) *If R^L has finite Goldie dimension as a right R^L-module, then R has finite Goldie dimension as a right R-module.*
(v) *If A is a subring of R^L such that R^L has Krull dimension α as a right A-module, then R has Krull dimension α as a right A-module.*

Proof: We proceed by induction on $\dim_K L$. If $\dim_K L = 1$, then we are done by Proposition 2.2. We may now assume that the result is true for all superalgebras whose dimension is less than $\dim_K L$. We can consider $\psi(L)$ to be acting on R, thus if ψ is not injective then $\dim_K \psi(L) < \dim_K L$ and we are done by the induction hypothesis. Therefore we may assume that L and $\psi(L)$ are isomorphic. However, by the argument above, $\psi(L)$ is nilpotent, thus we may assume that L is nilpotent.

Since L is nilpotent with $\dim_K(L) > 1$, L contains a proper superideal $I \neq 0$. The quotient superalgebra L/I acts on R^I with $R^L = (R^I)^{L/I}$. Since $\dim_K I < \dim_K(L)$ and $\dim_K L/I < \dim_K(L)$, we will make frequent use of the induction hypothesis.

For (i), R^I is a Noetherian right R^L-module finitely generated by a set $\{a_1, \dots, a_s\}$ and R is a Noetherian module over the right Noetherian ring R^I, generated over R^I by a set $\{b_1, \dots, b_t\}$. Therefore R is generated over the right Noetherian ring R^L by the finite set $\{b_i a_j\}_{i \leq t,\ j \leq s}$. Hence R is a Noetherian right R^L-module and so, R must be a right Noetherian ring. For (ii), since R^L is a right Artinian ring with unit, it is a right Noetherian ring. Therefore, by (i), R is finitely generated over R^L. However, a finitely generated module over an Artinian ring is Artinian, thus R is an Artinian right R^L-module and is therefore certainly a right Artinian ring. For (iii), the finite-dimensionality of R^L over K along with the induction hypothesis immediately implies that R^I is finite-dimensional over K. The same argument applied to I acting on R shows that R is also finite-dimensional over K. Part (iv) follows easily by using the induction hypothesis to go up from R^L to R^I and then from R^I to R.

Since L is nilpotent, the superideal I can be chosen such that I is one-dimensional, is spanned by a homogeneous element δ, and $[I, L] = 0$. For (v), we first apply the induction hypothesis to obtain $K\dim R^I{}_A = K\dim R^L{}_A$. Next, we can use Proposition 2.2(v) to conclude that $K\dim R_A = K\dim R^I{}_A$. As a result, it follows that $K\dim R_A = K\dim R^L{}_A$. ∎

We conclude this section with an observation which will be used in Sections 3 and 4. If δ is a nilpotent skew derivation of R and if $T \neq 0$ is a δ-stable subspace of R, then $T \cap R^{(\delta)} \neq 0$. By applying induction to $\dim_K L$, as in the proof of Theorem 2.3, it follows that

PROPOSITION 2.4: *Let R be a K-algebra acted on by a finite-dimensional Lie superalgebra L of nilpotent derivations and superderivations and let $T \neq 0$ be an L-stable subspace of R. Then $T \cap R^L \neq 0$.*

3. The general case

In order to prove the main result of this section, we need to combine the results of the previous section with some known results on group-graded rings having finite support. Most of the facts we need on group-graded rings appear in [CR] and can be summarized as

PROPOSITION 3.1: *Let S be a semiprime algebra graded by a group G with finite support and let S_1 denote the identity component.*

(i) *If S_1 is right Noetherian, then S is a Noetherian right S_1-module. In particular, S is right Noetherian and is a finitely generated right S_1-module.*
(ii) *If S_1 is right Artinian, then S is an Artinian right S_1-module. In particular, S is right Artinian and is a finitely generated right S_1-module.*
(iii) *If S_1 has finite Goldie dimension as a right S_1-module, then S has finite Goldie dimension as a right S_1-module.*
(iv) *If A is a subring of S_1 such that S_1 has Krull dimension α as a right A-module, then S has Krull dimension α as a right A-module.*

Proof: By [CR, Proposition 1.2], S_1 is semiprime. Then part (i) follows from [CR, Corollary 1.8] and part (ii) follows from [CR, Theorem 1.4]. Part (iii) follows from [CR, Proposition 1.5]. For part (iv), since S_1 is semiprime and has Krull dimension at most α as a right S_1-module, it follows that S_1 is right Goldie. Then, by [CR, Theorem 1.7], S is contained as a right S_1-module in a finite direct sum of copies of S_1. Therefore, S and S_1 have the same Krull dimension as right A-modules. ∎

In order to apply the results on group-graded rings, we will need to extend the ground field K. Suppose $A \subseteq B$ are K-algebras and let K' be a finite-dimensional field extension of K. If A is Noetherian, Artinian, of finite Goldie dimension, or of Krull dimension α as a right A-module, then the same holds for $A \otimes_K K'$ as a right $A \otimes_K K'$-module [MR, Chapter 10]. Similarly, if $B \otimes_K K'$ is finitely generated, of finite Goldie rank, or of Krull dimension at most α as a right $A \otimes_K K'$-module, then the same holds for B as a right A-module. As a result, to prove various results on the structure of B as a right A-module we may, if needed, extend the ground field K.

For the remainder of this section, we will assume that L is nilpotent. We now prove a technical lemma which shows how the nilpotence of L allows us to combine the results from Theorem 2.3 with those from Proposition 3.1.

LEMMA 3.2: *Let L be a finite-dimensional nilpotent Lie superalgebra acting on a K-algebra R such that if characteristic $K = p$ then L is restricted and if characteristic $K = 0$ then L acts on R as algebraic derivations and algebraic superderivations. Then there exists a finite-dimensional separable field extension $K' \supseteq K$ such that $L' = L \otimes_K K'$ acts on $R' = R \otimes_K K'$ and R' contains an L'-stable subalgebra B such that*

(i) $(R')^{L'} \subseteq B \subseteq R'$.
(ii) *B is the identity component of R' under the grading of R' by a group G with finite support.*
(iii) *The restriction of the action of L' to B is as nilpotent derivations and superderivations.*
(iv) *If R is semiprime then R' is semiprime.*

Proof: We must consider the characteristic p and characteristic 0 cases separately. If L is restricted in characteristic p, for any $n \geq 0$, let $Z_{(n)}$ denote the K-linear span of the set $\{z^{[p^n]}\colon z \in L_0 \text{ and } [z, L] = 0\}$. By the finite-dimensionality of L_0, there exists an $N \geq 0$ such that $Z_{(N)} = Z_{(N+1)}$. Letting $I = Z_{(N)}$, we observe that I is an abelian restricted ideal of L contained in L_0. Furthermore, by the choice of N, if $\{z_1, \ldots, z_s\}$ is a basis of I, then $\{{z_1}^{[p]}, \ldots, {z_s}^{[p]}\}$ is also a basis of I. Therefore by Hochschild's theorem on the semisimplicity of restricted enveloping algebras [H], the restricted enveloping algebra $u(I)$ is semisimple.

In [BC] the actions of commutative semisimple Hopf algebras are studied. It is shown that if H is a finite-dimensional commutative semisimple Hopf algebra acting on a K-algebra R, then there exists a finite-dimensional separable field extension K' of K such that $H \otimes_K K' = (K'G)^*$, the dual of a group algebra of a finite group G, and $R \otimes_K K'$ is graded by G with identity component $R^H \otimes K'$.

Let $R' = R \otimes_K K'$, $L' = L \otimes_K K'$, and $I' = I \otimes_K K'$. The K-linear action of L on R extends to a K'-linear action of L' on R' with $(R')^{L'} = R^L \otimes_K K'$. By the construction of I, the restriction of the derivations and superderivations of L to R^I are nilpotent K-linear transformations, hence the derivations and superderivations of L' are nilpotent K'-linear transformations when restricted to $R^I \otimes_K K'$. We now have the following chain of K'-algebras, $(R')^{L'} \subseteq R^I \otimes_K K' \subseteq R'$ where $R^I \otimes_K K'$ is the identity component of R' under the grading of a finite group and the action of L' restricted to $R^I \otimes_K K'$ is as nilpotent derivations and superderivations. Finally, it is shown in [BC, Lemma 3], that if K' is separable over K and if R is semiprime, then R' is also semiprime.

In the characteristic 0 case, we again want to find a finite-dimensional field extension $K' \supseteq K$ which allows us to decompose $R' = R \otimes_K K'$ as we did in characteristic p. Since $L = L_0 \oplus L_1$ is a nilpotent Lie superalgebra, L_0 is certainly a nilpotent Lie algebra. Now let $A = \{r \in R\colon d^n(r) = 0$, for all $d \in L_0$ where $n = n(r,d) \geq 1\}$; since L_0 is nilpotent, it follows from [B1, Lemma 1.6] that A is an L_0-stable K-subalgebra of R on which L_0 acts as nilpotent derivations. However, the argument used in the proof of [B1, Lemma 1.6] can be easily adapted to show that A is also L-stable where L acts on A as nilpotent derivations and superderivations.

If $\{x_1, \ldots, x_s\}$ is a K-basis of L_0, let K' be a finite-dimensional field extension of K which contains the eigenvalues of the action of each x_i on R. Then, since L_0 is nilpotent, $R' = R \otimes_K K'$ is graded with finite support by the set G of K'-linear maps from $L_0 \otimes_K K'$ to K'. We note that the identity component of R' under this grading is the set $B = \{r \in R'\colon d^n(r) = 0$, for all $d \in L_0 \otimes_K K'$ where $n = n(r,d) \geq 1\}$. L' acts on B as nilpotent derivations and superderivations and, since L_0 is nilpotent, B is equal to the set $A \otimes_K K'$.

Therefore we have a chain of K'-algebras $R^L \otimes_K K' = (R')^{L'} \subseteq B \subseteq R'$ as in the characteristic p case. Furthermore, since all extensions in characteristic 0 are separable, then R' is semiprime if R is. ∎

We are now in a position to prove the main result of this section.

THEOREM 3.3: *Let R be a semiprime K-algebra acted on by a finite-dimensional* *nilpotent Lie superalgebra L such that if characteristic $K = p$ then L is restricted and if characteristic $K = 0$ then L acts on R as algebraic derivations and algebraic superderivations.*

(i) *If R^L is right Noetherian, then R is a Noetherian right R^L-module. In particular, R is right Noetherian and is a finitely generated right R^L-module.*

(ii) *If R^L is right Artinian, then R is an Artinian right R^L-module. In particular, R is right Artinian and is a finitely generated right R^L-module.*

(iii) *If R^L is finite-dimensional over K then R is also finite-dimensional over K.*

(iv) *If R^L has finite Goldie dimension as a right R^L-module, then R has finite Goldie dimension as a right R-module.*

(v) *If R^L has Krull dimension α as a right R^L-module, then R has Krull dimension α as a right R^L-module. Thus R has Krull dimension at most α as a right R-module.*

Proof: Let K' be a separable extension of K as in Lemma 3.2; therefore we have the chain $R^L \otimes_K K' = (R')^{L'} \subseteq B \subseteq R' = R \otimes_K K'$. If R^L is Noetherian or Artinian as a right R^L-module, then so is $(R')^{L'}$ as a right $(R')^{L'}$-module. Since the action of L' on B is as nilpotent derivations and nilpotent superderivations, then by Theorem 2.3 (i) and (ii), B is finitely generated as a right $(R')^{L'}$-module. However, by Proposition 3.1 (i) and (ii), R' is finitely generated as a right B-module, thus R' is a finitely generated right $(R')^{L'}$-module. As a result, R is finitely generated as a right R^L-module. Parts (i) and (ii) now follow as finitely generated modules over Noetherian or Artinian rings are Noetherian or Artinian. For part (iii), if R^L is finite-dimensional over K then it is a right Artinian ring. Hence R is finitely generated over R^L and therefore must also be finite-dimensional over K.

If R^L has finite Goldie dimension as a right R^L-module then the same holds for $(R')^{L'}$ as a right $(R')^{L'}$-module and therefore, by Theorem 2.3 (iv), the same also holds for B as a right B-module. Thus, by Proposition 3.1 (iii), R' has finite Goldie dimension as a right R'-module and therefore the same is true for R as a right R-module, thereby proving part (iv). $(R')^{L'}$ is a finite direct sum of copies of R^L as a right R^L-module, thus if R^L has Krull dimension α as a right R^L-module then so does $(R')^{L'}$. By Theorem 2.3(v), B also has Krull dimension α as a right R^L-module. Thus, by Proposition 3.1(iv), R' has Krull dimension α as a right R^L-module. Since $R^L \subseteq R \subseteq R'$, it is clear that R has Krull dimension α as a right R^L-module and has Krull dimension at most α as a right R-module.

We conclude this section by extending [B1, Theorem 1.8] on the existence of invariants of the action of Lie algebras to Lie superalgebras.

THEOREM 3.4: *Let R be a K-algebra acted on by a finite-dimensional nilpotent Lie superalgebra L such that if characteristic $K = p$ then L is restricted and if characteristic $K = 0$ then L acts on R as algebraic derivations and algebraic superderivations. If A is a non-nilpotent L-stable subalgebra of R then $A \cap R^L \neq 0$.*

Proof: As in Lemma 3.2, we extend the ground field and let $R' = R \otimes_K K'$, $L' = L \otimes_K K'$, $A' = A \otimes_K K'$, and we also let B be as in Lemma 3.2. Since A is L-stable, A' is a graded subspace of the graded algebra $R' = R \otimes_K K'$. By [CR, Proposition 1.2], since A' is now non-nilpotent and graded, $A' \cap B \neq 0$. Since $A' \cap B$ is a non-zero L-stable subspace of B it follows, by Proposition 2.4, that $A' \cap R' \neq 0$. However, $A' \cap R' = (A \cap R^L) \otimes_K K'$, thereby proving the result. ∎

4. Central rings of invariants and polynomial identities

In this section we prove some further going-up theorems from R^L to R, however the techniques used will be somewhat different from those in Sections 2 and 3. In [BCF, Theorem 2.8], it is shown that if R is prime and if L is a nilpotent Lie algebra acting on R in a certain "finite" manner where R^L is central, then R satisfies a polynomial identity. We will extend this result in two ways: (i) for prime rings we extend the result from the action of Lie algebras to the action of Lie superalgebras and (ii) for the action of Lie algebras we extend the result from prime rings to semiprime rings. At this point, we must be more precise as to the meaning of L acting on R in a "finite" manner.

Suppose L is a finite-dimensional Lie superalgebra acting on R such that if characteristic $K = p$ then L is restricted and if characteristic $K = 0$ then L acts on R as algebraic derivations and algebraic superderivations. Then it is clear that the image of the universal enveloping algebra $U(L)$ in $\mathrm{End}_K(R)$ is finite-dimensional. Furthermore, if L is a Lie superalgebra but not a Lie algebra, then there exists a group G of order 2 such that the skew group ring $U(L) * G$ is a Hopf algebra which acts on R and the image of $U(L)*G$ is also finite-dimensional in $\mathrm{End}_K(R)$. To unify the two cases, if L is a Lie algebra then we let $H = U(L)$ and if L is a Lie superalgebra with $L_1 \neq 0$, we let $H = U(L) * G$. Then we say that H acts **finitely of dimension** N if the image of H in $\mathrm{End}_K(R)$ has dimension N. In the restricted case let $u(L)$ be the restricted enveloping algebra of L; then N is bounded by $\dim_K u(L)$ when L is a Lie algebra and N is bounded by $2 \cdot \dim_K u(L)$ when L is a Lie superalgebra with $L_1 \neq 0$.

LEMMA 4.1: *Let R be a semiprime algebra on which $H = U(L)$ or $H = U(L)*G$ acts finitely. If $\lambda \neq 0$ is an H-stable left ideal of R and $R^L \subseteq Z(R)$, then $\lambda \cap R^H \neq 0$.*

Proof: If $H = U(L)$, then $R^L = R^H$ and the result is a special case of Theorem 3.4. Therefore, we will assume that $L_1 \neq 0$ and $H = U(L) * G$. If $\delta \in L_0 \cup L_1$ and $a \in R$, then either $\delta(\sigma(a)) = \sigma(\delta(a))$ or $\delta(\sigma(a)) = -\sigma(\delta(a))$. Therefore if $a \in R^L$, then $\sigma(a) \in R^L$, hence σ acts on R^L. By Theorem 3.4, $\lambda \cap R^L \neq 0$ and if $0 \neq b \in \lambda \cap R^L$, then $b + \sigma(b) \in \lambda \cap R^H$. As a result, if $\sigma(b) \neq -b$ then $\lambda \cap R^H \neq 0$. However, if $\sigma(b) = -b$ then $\sigma(b^2) = b^2 \in \lambda \cap R^H$. Since b is central, we have $b^2 \neq 0$, and hence $\lambda \cap R^H \neq 0$. ∎

We now handle the important special case where R^H is a field.

LEMMA 4.2: *Let R be a semiprime algebra on which $H = U(L)$ or $H = U(L)*G$ acts finitely of dimension N. If $R^L \subseteq Z(R)$ and if R^H is a field, then R satisfies a polynomial identity of degree at most $2[\sqrt{N}\,]$, where $[\sqrt{N}\,]$ is the greatest integer in $\sqrt{N}$.*

Proof: When we let the smash product $R\#H$ act on the ring R, the $R\#H$-submodules of R are the H-stable left ideals of R. Since all the elements of R^H are invertible in R, it follows by Lemma 4.1, that R is an irreducible left $R\#H$-module. If $H = U(L)$ then clearly $R^H = R^L$. On the other hand, if $H = U(L)*G$, we note that R^L is a semiprime ring with fixed ring R^H under the action of G. Since R^H is a field and R^L has no $|G|$-torsion, it follows that R^L is finite-dimensional over R^H. Thus regardless of whether $H = U(L)$ or $U(L) * G$, R^L is a right Artinian ring. Therefore, by Theorem 3.3(ii), R is finitely generated as a right R^L-module, hence R is also finite-dimensional over R^H. Thus R has finite Goldie rank as a left R-module. As a result, we can apply [BCF, Theorem 2.2] which says that if A is a left H-module algebra such that $A\#H$ acts irreducibly on A, A has finite left Goldie rank, and H acts finitely of dimension N, then the dimension of A as a left vector space over A^H is at most N.

Therefore R has dimension at most N over the central subfield R^H and it follows by basic facts on the polynomial identities of semiprime rings, that R satisfies a standard identity of degree at most $2[\sqrt{N}\,]$. ∎

We can now extend [BCF, Theorem 2.8] to the action of Lie superalgebras. The proof below is written for Lie superalgebras, but it can easily be specialized to Lie algebras by ignoring the presence of G and the automorphism σ.

THEOREM 4.3: *Let R be a prime K-algebra acted on by a finite-dimensional nilpotent Lie superalgebra L such that if characteristic $K = p$ then L is restricted and if characteristic $K = 0$ then L acts on R as algebraic derivations and algebraic superderivations. If R^L is central, then R satisfies a polynomial identity.*

Proof: The Hopf algebra $U(L) * G$ acts finitely on R with $R^H \subseteq R^L \subseteq Z(R)$. Next we localize R at the nonzero elements of R^H to obtain a new prime H-module algebra S. If $s \in S^L$, then $s = r\alpha^{-1}$, where $r \in R$ and $\alpha \in R^H$. If $\delta \in L_0 \cup L_1$ then $0 = \delta(s) = \delta(r\alpha^{-1}) = \delta(r)\alpha^{-1}$, hence $r \in R^L$ and so, S^L is central. Furthermore, if $s = r\alpha^{-1} \in S^H$ then $r\alpha^{-1} = s = \sigma(s) = \sigma(r\alpha^{-1}) = \sigma(r)\alpha^{-1}$. Hence $r \in R^H$ and thus S^H is the quotient field of R^H. As a result, we can apply Lemma 4.2 to conclude that S satisfies a polynomial identity and thus R also satisfies a polynomial identity. ∎

We now assume that L is a Lie algebra and we will extend the result in [BCF] to semiprime rings. The key is to try to reduce to the case where R^L is a field and then apply Lemma 4.2. In order to do this, we need to extend the action of L to various other K-algebras. If Q is the symmetric Martindale quotient ring of R, then the action of L always extends uniquely to Q. Q is semiprime and its center C, known as the extended center of R, is von Neumann regular. Furthermore, if a derivation of R is algebraic then its extension to Q satisfies the same polynomial. Therefore the hypothesis that L acts finitely of dimension N on R also extends to the action of L on Q. For the remainder of this section, we will assume that L is a finite-dimensional nilpotent Lie algebra acting finitely on the semiprime ring R with $R^L \subseteq Z(R)$.

LEMMA 4.4: $Q^L \subseteq C$.

Proof: Let $q \in Q^L$ and let I be an essential ideal of R such that $Iq, qI \subseteq R$. It is clear that $I^L q \subseteq R^L$. It suffices to show that $[I, q] = 0$ and, to this end, we note that $0 = [I, I^L q] = I^L[I, q]$. Thus if we let $J = \{r \in R\colon I^L r = 0\}$, it now suffices to show that $J = 0$. Since all the derivations of L are algebraic, it follows by [B1, Proposition 1.12] that all nonzero ideals of R contain a nonzero L-stable ideal. If $J \neq 0$, then $I \cap J \neq 0$ and therefore $I \cap J$ contains a nonzero L-stable ideal of R. Thus, by Lemma 4.1, there exists a nonzero $a \in (I \cap J)^L$. As a result, $a^2 \in I^L J = 0$. However, this is a contradiction as a is central in a semiprime ring. ∎

The center of any ring is invariant under all derivations of the ring. However, this need not be the case for skew derivations. In fact, in Section 5 we will see an example of a Lie superalgebra acting on a semiprime ring whose center is not L-stable. We do not know if the results in this section can be extended to the action of Lie superalgebras on semiprime rings. However, the remaining arguments in this section do not apply to the action of Lie superalgebras, since they require that C be L-stable.

LEMMA 4.5: *Q^L is a von Neumann regular ring.*

Proof: If $a \in Q^L$, we need to find some $b \in Q^L$ such that $a^2 b = a$. Since C is von Neumann regular, we know that there exists some $c \in C$ with $a^2 c = a$. We will use c to construct an appropriate $b \in Q^L$. Let $e = ac$, then $e^2 = (ac)^2 = (a^2 c)c = ac = e$. Thus e is a central idempotent in Q and as a result, $\delta(e) = 0$, for all $\delta \in L$. Therefore $0 = \delta(ac) = a\delta(c)$ and multiplying this equation by e yields $0 = a\delta(c)e = a\delta(ce)$. However $a = ae \in Ce$ and so, a is invertible in Ce. Therefore since $\delta(ce) \in Ce$, we have $\delta(ce) = 0$. Furthermore, $a^2(ce) = (a^2 c)e = ae = a$ and thus $b = ce$ is the desired element of Q^L. ∎

Since Q^L is central, if M is any maximal ideal of Q^L then we can localize Q at M to form the K-algebra Q_M. Note that the elements of Q which become 0 in Q_M are precisely those which are annihilated by some element of $Q^L - M$. Clearly, the action of L on Q determines a unique action of L on Q_M.

LEMMA 4.6: *$(Q^L)_M = (Q_M)^L$.*

Proof: The inclusion $(Q^L)_M \subseteq (Q_M)^L$ is clear. For the reverse inclusion, let $q\alpha^{-1} \in (Q_M)^L$. Thus for all $\delta \in L$, $\delta(q\alpha^{-1}) = 0$ in Q_M. Therefore if $\{\delta_1, \ldots, \delta_n\}$ is a basis for L, then for every $i \leq n$ there exists $m_i \in Q^L - M$ such that $m_i \delta(q)\alpha^{-1} = 0$ in Q. Let $m = \prod_1^n m_i$, then $m \in Q^L - M$ and for all $\delta \in L$, $\delta(mq) = m\delta(q) = 0$. As a result, $mq \in Q^L$ and $q\alpha^{-1} = (mq)(m\alpha)^{-1} \in (Q^L)_M$. ∎

In general, the localization of a semiprime ring at a maximal ideal of a central subring need not be semiprime. However, in our specialized situation we have

LEMMA 4.7: *Q_M is semiprime.*

Proof: Suppose $a \in Q_M$ such that $aQ_Ma = 0$, it suffices to show that $a = 0$ in Q_M. Then $a = q\alpha^{-1}$, for some $q \in Q$ and $\alpha \in Q^L - M$ and $qQq = 0$ in Q_M. If $f \in \mathrm{Ann}_C(qQq)$, then $(fq)Q(fq) = 0$. Therefore, since Q is semiprime, $fq = 0$, hence $\mathrm{Ann}_C(qQq) = \mathrm{Ann}_C(q)$. In addition, Q is a complete C-module [Kh, Lemma 1.6.14] and qQq is a closed additive subgroup of Q. Thus by [Kh, Lemma 1.6.26], there exists some $s \in Q$ such that $\mathrm{Ann}_C(qsq) = \mathrm{Ann}_C(qQq)$. Since $qsq = 0$ in Q_M, there exists an $m \in Q^L - M$ such that $mqsq = 0$. However, $\mathrm{Ann}_C(qsq) = \mathrm{Ann}_C(q)$, hence $mq = 0$. As a result, $q = 0$ in Q_M and so, $a = 0$ in Q_M, thereby proving the result. ∎

We can now put the pieces together to prove the main results of this section. We state the characteristic p and characteristic 0 cases separately, since the conclusion in the characteristic 0 case will be much stronger.

THEOREM 4.8: *Let R be a semiprime K-algebra acted on by a finite-dimensional restricted nilpotent Lie algebra L, where K has characteristic p. If R^L is central, then R satisfies a polynomial identity of degree $2[\sqrt{p^{\dim_K L}}\,]$.*

Proof: We extend the action of L to Q and by Lemma 4.4, $Q^L \subseteq C$. Therefore Q embeds in $\prod_M Q_M$, where the product is taken over the maximal ideals of Q^L. Since $R \subseteq Q$, it suffices to show that each Q_M satisfies the standard identity of degree $2[\sqrt{p^{\dim_K L}}\,]$. If M is any maximal ideal of Q^L, then L acts on Q_M. By Lemmas 4.4 and 4.5, Q^L is a von Neumann regular ring contained in C, therefore [P, Lemma 18.1(i)] $(Q^L)_M$ is a central subfield of Q_M. However by Lemma 4.6, $(Q^L)_M = (Q_M)^L$, thus $(Q_M)^L$ is now a central subfield of Q_M. By Lemma 4.7, Q_M is semiprime and we are therefore in a position to apply the special case of Lemma 4.2, where $H = U(L)$ and $|G| = 1$. As a result, Q_M satisfies the standard identity of degree $2[\sqrt{N}\,]$, where $N = \dim_K u(L) = p^{\dim_K L}$. Thus R also satisfies the standard identity of degree $2[\sqrt{p^{\dim_K L}}\,]$. ∎

All algebraic derivations of a semiprime ring R of characteristic 0 become inner when extended to Q. As a result, studying the invariants of algebraic derivations of semiprime rings in characteristic 0 reduces to studying centralizers of certain subsets of Q. Thus we can now prove

THEOREM 4.9: *Let R be a semiprime K-algebra with characteristic $K = 0$ and let L be a finite-dimensional nilpotent Lie algebra which acts on R as algebraic derivations. If R^L is central then R is commutative and the action of L on R is trivial.*

Proof: Since all derivations of L become inner in Q, it suffices to show that Q is commutative. Following the argument used in the proof of Theorem 4.8, Q_M satisfies a polynomial identity, for every maximal ideal $M \subseteq Q^L$. Therefore it now suffices to show that Q_M is commutative. Q_M is a semiprime ring satisfying a polynomial identity whose center is a field. Therefore, as in the proof of Theorem 4.8, Q_M is a central simple algebra. Tensoring by the algebraic closure of K, we may assume that $Q_M \cong K_t$, the $t \times t$ matrices over K. Without loss, we may identify L with its image in $\mathrm{End}_K(Q_M)$. Therefore, by Lemmas 4.4 and 4.6, it will be enough to show that $L = 0$. If $L \neq 0$ then, since L is nilpotent, it follows that $Z(L) \neq 0$. In this case, let $0 \neq \delta \in Z(L)$ and suppose the derivation δ is induced by $a \in Q_M$. If $d \in L$, then d is induced by some $b \in Q_M$ and $[\delta, d] = 0$ in L. Since the derivation $[\delta, d]$ is induced by $[a, b] \in Q_M$, we have that $[a, b] \in Z(Q_M)$. Furthermore, since K has characteristic 0, no nonzero central elements of K_t can have trace equal to 0. However, $[a, b]$ is a commutator in Q_M, hence it has trace 0 and so, $[a, b] = 0$. As a result a commutes with all the elements of Q_M which induce derivations from L, hence $a \in (Q_M)^L \subseteq Z(Q_M)$. Thus $\delta = 0$ in $\mathrm{End}_K(Q_M)$, a contradiction. Therefore the action of L on Q_M is trivial, hence Q_M is commutative and the proof is complete. ∎

In light of Theorem 4.9, it is reasonable to wonder if the conclusion of Theorem 4.8 can be strengthened to R being commutative. Similarly, it is reasonable to wonder in the characteristic 0 case, if the conclusion of Theorem 4.3 can be strengthened to R being commutative. However, in the next section we will see examples showing that the conclusions of these theorems cannot be strengthened to commutativity. In fact, we will see that the bound on the degree of the polynomial identity in Theorem 4.8 is best possible.

5. Counterexamples

In this section we provide counterexamples to various questions which arise in view of the results in Sections 2, 3, and 4. Let d be an algebraic derivation of a semiprime ring R. In [GM] it is shown that if R is left Goldie then $R^{(d)}$ is left

Goldie, and in [G] it is shown that if R is left Artinian then $R^{(d)}$ is left Artinian. In the next example, we show that the analogous result does not hold for left Noetherian rings.

Recall that when d is a skew derivation, d is a right $R^{(d)}$-module map, but not necessarily a left $R^{(d)}$-module map. As a result, all of the theorems in Sections 2 and 3 are stated for right $R^{(d)}$-modules. If $R^{(d)}$ is well behaved under the action of the automorphism τ which corresponds to d, then similar results also hold on the left. In particular, if d is a derivation, then $\tau = 1$ and d is both a left and right $R^{(d)}$-module map. Therefore, if L is a Lie algebra, the results in Sections 2 and 3 are equally valid for left R^L-modules. For convenience, the examples in this section will sometimes be stated in terms of left R^L-modules. However, when L is a Lie algebra, these examples could easily be reworked in terms of right R^L-modules.

Example 5.1: A prime left Noetherian ring R of arbitrary characteristic with a nilpotent derivation δ such that $R^{(\delta)}$ is not left Noetherian.

Let $A = K(t_0, t_1, \dots)$ be the rational functions over the ground field K and let τ be the injective homomorphism of A such that $\tau(t_i) = t_{i+1}$, for all $i \geq 0$. Now let B be the skew polynomial ring $A[x;\tau]$ where $xa = \tau(a)x$, for all $a \in A$, and every element of B is of the form $\sum a_i x^i$, $a_i \in A$. Next, let $R = B_2$, the 2×2 matrices over B. Since B is a left Noetherian domain, R is a prime left Noetherian ring. Let δ be the inner derivation of R defined as commutation by the element $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$. Clearly $\delta^3 = 0$ and if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R^{(\delta)}$ a direct calculation shows that $ax = xd$ and $c = 0$. Thus if $a = \sum a_i x^i$ and $d = \sum d_i x^i$, where $a_i, d_i \in A$, it follows that $\sum a_i x^{i+1} = ax = xd = x \sum d_i x^i = \sum \tau(d_i) x^{i+1}$. As a result, after extending τ to all of B by mapping x to itself, we have $a = \tau(d) \in \tau(B) = k(t_1, t_2, \dots)[x;\tau]$. Next, for every natural number i, let $V_i = \tau(B) + \tau(B)t_0 + \cdots + \tau(B){t_0}^i$. Each V_i is a left $\tau(B)$-submodule of B and $V_i \subsetneq V_{i+1}$ as ${t_0}^{i+1} \in V_{i+1} - V_i$. Now let $T_i = \begin{pmatrix} 0 & V_i \\ 0 & 0 \end{pmatrix}$; since $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R^{(\delta)}$ implies that $a \in \tau(B)$, it follows that each T_i is a left ideal of $R^{(\delta)}$. Since $T_1 \subsetneq T_2 \subsetneq T_3 \subsetneq \cdots$ is an infinite ascending chain of left ideals of $R^{(\delta)}$, $R^{(\delta)}$ is not left Noetherian. It is also interesting to note that $R = R^{(\delta)}e_{21} + R^{(\delta)}e_{22}$, hence R is a finitely generated left $R^{(\delta)}$-module. ∎

In light of Example 5.1, it would be interesting to know if $R^{(\delta)}$ must be left

Noetherian when R is prime and both left and right Noetherian. The results of [GM] and [G] mentioned above can be viewed as going-down results for the action of 1-dimensional Lie algebras. However, in the next example we show that these results cannot be extended even to 2-dimensional abelian Lie algebras.

Example 5.2: A simple Artinian ring R of arbitrary characteristic acted on by a 2-dimensional abelian Lie algebra L of nilpotent derivations such that R^L does not have finite left Goldie rank as a left R^L-module. In particular, R^L is neither left Artinian nor left Goldie.

Let D be any division ring containing an element x, such that D has infinite dimension as a left vector space over A, the centralizer of x in D. Let $\{a_1, a_2, \dots\}$ be elements of D which are left independent over A, hence $Aa_1 \oplus Aa_2 \oplus \cdots$ is an infinite direct sum of left A-submodules of D. Now let $R = D_2$, the 2×2 matrices over D, hence R is a simple Artinian ring. Next, let δ_1 and δ_2 be the inner derivations of R induced by $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Clearly δ_1 and δ_2 span a 2-dimensional abelian Lie algebra L of derivations such that ${\delta_1}^3 = {\delta_2}^3 = 0$. As in the previous example, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R^L$, then $ax = xd$, $a = d$, and $c = 0$. Hence, a is an element of A. For every $i \geq 1$, let $T_i = \begin{pmatrix} 0 & Aa_i \\ 0 & 0 \end{pmatrix}$, then each T_i is a left ideal of R^L. As a result, $T_1 \oplus T_2 \oplus \cdots$, is an infinite direct sum of left ideals of R^L. Thus R^L has infinite Goldie rank as a left R^L-module and so, R^L is neither left Artinian nor left Goldie. ∎

In the Bergman-Kharchenko example mentioned in Section 1, R is a matrix ring over a noncommutative free algebra. It is reasonable to wonder if similar examples hold for rings satisfying various chain conditions. In the next example, we modify the Bergman-Kharchenko example so that R is a matrix ring over a left Ore domain.

Example 5.3: A prime left Goldie ring R of arbitrary characteristic acted on finitely by a solvable 3-dimensional Lie algebra L such that R contains an L-stable ideal $I \neq 0$ with $I \cap R^L = 0$.

We slightly modify the ring used in Example 5.1 as we let $A = K[t_0, t_1, \dots]$ be the commutative polynomial ring over the ground field K and let τ be the injective homomorphism of A such that $\tau(t_i) = t_{i+1}$, for all $i \geq 0$. Now let B be the skew polynomial ring $A[x; \tau]$ where $xa = \tau(a)x$, for all $a \in A$, and every element of B is of the form $\sum a_i x^i$, $a_i \in A$. Next, let $R = B_2$, the 2×2 matrices over B. Since B is a left Ore domain, R is a prime left Goldie ring. Let L be spanned by the three inner derivations of R induced by $f_1 = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$, $f_2 = \begin{pmatrix} 0 & t_0 \\ 0 & 0 \end{pmatrix}$, and $f_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. As in the Bergman-Kharchenko example, L is a solvable 3-dimensional Lie algebra such that in characteristic p, L is restricted and in characteristic 0, L acts as algebraic derivations. Let I be the ideal $R\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$; if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I$ commutes with both f_1 and f_2, then, we have $ax = xd$, $at_0 = t_0 d$, and $c = 0$. Since $ax = xd$ it follows, as in Example 5.1, that if $a = \sum a_i x^i$ and $d = \sum d_i x^i$, with $a_i, d_i \in A$, then $\sum a_i x^{i+1} = ax = xd = x \sum d_i x^i = \sum \tau(d_i) x^{i+1}$. Thus $a = \sum \tau(d_i) x^i$. Since $at_0 = t_0 d$, we also have $\sum t_i \tau(d_i) x^i = \sum \tau(d_i) x^i t_0 = at_0 = t_0 d = \sum t_0 d_i x^i$. Thus for all $i \geq 1$, $t_i \tau(d_i) = t_0 d_i$. However this is impossible, unless $d_i = 0$, since the largest subscript of the t's in $t_i \tau(d_i)$ exceeds the largest subscript appearing in $t_0 d_i$. As a result, $a = d = c = 0$. Finally, if $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in I$ also commutes with f_3, then $b = 0$. Hence $I \cap R^L = 0$. ∎

The results in Section 3 on the finite generation of R as a R^L-module all assume that R^L satisfies a chain condition. In the next example, we show that if there are no hypotheses on R^L, then R need not be finitely generated over R^L even if R is prime and L is 1-dimensional.

Example 5.4: A prime ring R of arbitrary characteristic with a nilpotent derivation δ such that R is not finitely generated as a left $R^{(\delta)}$-module.

Let $A = K[x, y]$ be the free algebra in 2 noncommuting variables over the field K and let R be the prime ring A_2, the 2×2 matrices over A. Let δ be the inner derivation induced by $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$. As before, $\delta^3 = 0$ and if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R^{(\delta)}$ then $ax = xd$ and $c = 0$. Letting $B = K[x, y]x + K$ it follows that $d \in B$. B is a subring of A and A is not finitely generated as a left B-module, for if $\{d_1, d_2, \ldots, d_n\} \subset A$ then $Bd_1 + Bd_2 + \cdots + Bd_n$ can only contain a finite number of y^i. Now suppose $r_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in R$, for $i \leq n$. Then if $\begin{pmatrix} e & f \\ g & h \end{pmatrix} \in R^{(\delta)} r_1 + R^{(\delta)} r_2 + \cdots + R^{(\delta)} r_n$, it follows that $h \in Bd_1 + Bd_2 + \cdots + Bd_n$. However, since $Bd_1 + Bd_2 + \cdots + Bd_n \neq A$, R is not finitely generated as a left $R^{(\delta)}$-module. ∎

If R is a simple ring of characteristic 0 satisfying a polynomial identity, then all central commutators must be zero. This fact was crucial in the proof of Theorem

4.9, where we showed that R must be commutative. However, in the characteristic p case a simple ring satisfying a polynomial identity can have elements a, b such that $ab - ba = 1$. We use this fact to now show that Theorem 4.9 cannot be extended to the characteristic p case.

Example 5.5: A simple ring R of characteristic p acted on by a 2-dimensional restricted abelian Lie algebra L of nilpotent derivations such that R^L is central, but R is not commutative.

Let K be a field of characteristic p and let R be the simple ring K_p, the $p \times p$ matrices over K. Next let

$$a = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

and

$$b = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & & \\ 0 & 0 & 0 & \dots & p-1 & 0 \end{pmatrix},$$

and we let L be the Lie algebra spanned by the inner derivations induced by a and b. It is easily checked that $ab - ba = 1, a^p = 0$, and $b^p = 0$, therefore L is a 2-dimensional restricted abelian Lie algebra consisting of derivations all of whose pth power is 0. If T is the subalgebra of R generated by a and b, we claim that $T = R$. To this end, we first note that a^{p-1} and b^{p-1} are nonzero scalar multiples of the matrix units e_{1p} and e_{p1}. Thus $e_{11} \in T$. In addition, $b^{i-1}e_{11}$ and $e_{11}a^{j-1}$ are nonzero scalar multiples of e_{i1} and e_{1j}, for $2 \leq i, j \leq p$. As a result, T clearly contains all the matrix units of R and so, $T = R$. However, R^L is the centralizer in R of T, hence R^L is central even though R is not commutative. ∎

Not only does Example 5.5 show that R need not be commutative, but we can now adapt Example 5.5 to show that the bound on the degree of the polynomial identity in Theorem 4.8 is best possible.

Example 5.6: A simple ring S of characteristic p acted on by a $2n$-dimensional restricted abelian Lie algebra L of nilpotent derivations such that S^L is central, but the smallest degree of a polynomial identity satisfied by S is $2[\sqrt{p^{\dim_K L}}\,] = 2p^n$.

Let R, a, and b be as in the previous example. For any $n \geq 1$, let $S = R \otimes R \otimes \cdots \otimes R$, the tensor product of n copies of R, and then let L be the Lie algebra of inner derivations of S induced by the $2n$ elements $\{a \otimes 1 \otimes \cdots \otimes 1,\ 1 \otimes a \otimes \cdots \otimes 1, \ldots, 1 \otimes 1 \otimes \cdots \otimes a,\ b \otimes 1 \otimes \cdots \otimes 1,\ 1 \otimes b \otimes \cdots \otimes 1, \ldots, 1 \otimes 1 \otimes \cdots \otimes b\}$. It is now easy to see that $S \cong K_{p^n}$, $S^L \cong K$, and L is a restricted abelian $2n$-dimensional Lie algebra of nilpotent derivations. However, the smallest degree of a polynomial identity satisfied by K_{p^n} is $2p^n$. ∎

In light of Theorem 4.9, it is reasonable to wonder when R is prime of characteristic 0 and L is a Lie superalgebra, whether Theorem 4.3 can be generalized to show that R is commutative. However, in the next example we show that this is not the case.

Example 5.7: A simple ring R of characteristic 0 acted on by a 2-dimensional abelian Lie superalgebra L of nilpotent superderivations such that R^L is central, but R is not commutative.

Let K be a field of characteristic 0 and let R be the simple ring K_2, the 2×2 matrices over K. Let σ be the inner automorphism of order 2 of R induced by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and let δ_1 and δ_2 be the σ-derivations of R defined as $\delta_1(r) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} r - \sigma(r) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\delta_2(r) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} r - \sigma(r) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, for all $r \in R$. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R$, then explicit formulas for σ, δ_1, and δ_2 are $\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$, $\delta_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} b-c & a-d \\ a-d & b-c \end{pmatrix}$, and $\delta_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} b+c & d-a \\ a-d & b+c \end{pmatrix}$. Since the matrices $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are negated by σ, it follows that $\delta_1 \sigma = -\sigma \delta_1$ and $\delta_2 \sigma = -\sigma \delta_2$. Furthermore, it is not hard to check that $\delta_1 \delta_2 = -\delta_2 \delta_1$ and ${\delta_1}^2 = {\delta_2}^2 = 0$. As a result, if L is spanned by the σ-derivations δ_1 and δ_2 then L is a 2-dimensional abelian Lie superalgebra of nilpotent superderivations with $L = L_1$. It is also clear that R^L is central even though R is not commutative. ∎

In Section 4, we mentioned that when a Lie superalgebra L acts on a semiprime ring R, the center of R need not be L-stable. We conclude this paper with such an example.

Example 5.8: A semiprime ring R of arbitrary characteristic acted on by a 1-dimensional Lie superalgebra L spanned by a superderivation δ with $\delta^2 = 0$ such that the center of R is not L-stable.

Let S be any semiprime algebra with an element $a \notin Z(S)$ such that $a^2 \in Z(S)$. Next let $R = S \oplus S$; clearly R is also semiprime. Let σ and δ be defined as $\sigma(x,y) = (y,x)$ and $\delta(x,y) = (ax-ya, xa-ay)$, for all $(x,y) \in R$. It is clear that σ is an automorphism of R of order 2 and $\delta\sigma = -\sigma\delta$. Furthermore, it can be checked that δ is a σ-derivation of R such that $\delta^2 = 0$. Hence δ spans a 1-dimensional Lie superalgebra L acting on R, where $L = L_1$. Certainly $(1,0) \in Z(R)$, however $\delta(1,0) = (a,a) \notin Z(R)$. Thus the center of R is not L-stable. ∎

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